Final Examination, 10 May 1999
SM311O (Spring 1999)

The following formulas may be useful to you:

$$a)\ \oint_C \mathbf{v}\cdot d\mathbf{r} = \int\int_S \nabla\times\mathbf{v}\cdot d\mathbf{A},$$

$$b)\ \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \nabla\mathbf{v}\cdot\mathbf{v}\right) = -\nabla p + \mu\Delta\mathbf{v} + \rho\mathbf{F}, \qquad \text{div } \mathbf{v} = 0.$$

$$c)\quad -fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + A_V\frac{\partial^2 u}{\partial z^2}, \quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A_V\frac{\partial^2 v}{\partial z^2}, \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

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1. (a) Let $\mathbf{v} = \langle xy, a\sin(zy), z\sqrt{x}\rangle$ where a ia constant. Determine a so that the divergence of $\mathbf{v}$ vanishes at the point $P = (4, -1, \frac{\pi}{6})$.
   (b) Let $\mathbf{v} = \langle y^2, -x^2, 0\rangle$. Find the curl of $\mathbf{v}$. Is this flow irrotational anywhere?
   (c) Prove the identity $\nabla\times\nabla\phi = \mathbf{0}$ if ϕ is an arbitrary function of x, y, and z.
2. Verify by direct differentiation if
   (a) $u(z) = e^{2z}\cos 2z$ is a solution of $u'''' + a^2 u = 0$ for any a.
   (b) $u(x,y) = \sin x\cos 2y$ is an eigenfunction of the Laplace operator $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. What is the eigenvalue?
3. (a) Give a parametrization for the plane that passes through the points $(-1,1,0)$, $(0,2,2)$, and $(1,0,1)$.
   (b) Find a unit normal vector to the surface of the upper hemisphere of the Earth at the point whose longitude and latitude are 60 and 30 degrees, respectively.
4. (a) The function $\phi(x,y) = ax^2y^2 - by^2 + x$ is the potential for a velocity vector field $\mathbf{v}$. Determine the values of a and b so that the velocity of the particle located at $(2,-1)$ is zero.
   (b) The function $\psi(x,y) = ax^2 + xy + by^2$ is the stream function of a velocity field $\mathbf{v}$. Find a and b so that the velocity of the particle located at $(1,1)$ has magnitude $\frac{1}{2}$.
5. (a) Consider the velocity field $\mathbf{v} = x^2z\mathbf{k}$. Determine the flux of this fluid through the following two surfaces:
      i. a disk of radius 1 in the xy-plane and centered at the origin.
      ii. a disk of radius 1 in the plane $z = 1$ and centered at the origin.
   (b) Compute the flux of vorticity of $\mathbf{v} = y^2\mathbf{i}$ through the surface of the upper hemisphere of a sphere of radius 2 centered at the origin (Hint: Use the Stokes Theorem).

6. Consider the following heat conduction problem:

$$u_t = 7u_{xx}, \qquad u(0,t) = u(3,t) = 0, \qquad u(x,0) = x(3-x).$$

   (a) Use separation of variables and find the solution to this problem. Clearly indicate the process of separation of variables and the Fourier Series method used in obtaining this solution.

   (b) Use the first nonzero term of the above solution and estimate how long it takes for the temperature at $x = 1.5$ to reach 50 per cent of its original value.

7. Let $\mathbf{v} = \langle x^2 + y^2, 2xy\rangle$ be the velocity field of a fluid. Compute the acceleration $\mathbf{a}$ of this flow. Does $\mathbf{a}$ have a potential p? If yes, find it.

8. Let $\boldsymbol{\Omega}$ stand for the angular velocity of our planet.

   (a) Noting that our planet rotates once every 24 hours, compute Ω where $\boldsymbol{\Omega} = \langle 0, 0, \Omega\rangle$. What are the units of Ω?

   (b) Use this value of $\boldsymbol{\Omega}$ and estimate the values in the centripetal acceleration $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ where $\mathbf{r}$ is the position vector to a typical point on the surface of the Earth. Assume that the radius of the Earth is 6000 kilometers.

9. Consider an incompressible fluid occupying the basin

$$D = \{(x,y,z) | 0 \le z \le H\}.$$

   Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be the velocity field of a motion generated in D. Suppose that we have been able to determine that

$$v_1(x,y,z) = x^2y^2, \quad v_2(x,y,z) = -3x^2z,$$

   but have only succeeded in measuring v_3 at the bottom of the basin and that this value is

$$v_3(x,y,0) = x + y.$$

   Determine v_3 everywhere in D. (Hint: What does incompressibility mean **mathematically**?)

10. A flow is called geostrophic if the velocity $\mathbf{v} = \langle u(x,y), v(x,y)\rangle$ and the pressure gradient ∇p are related by

$$(*) \qquad -fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

   where ρ, a constant, is the density of the fluid, and f is the coriolis parameter.

   (a) Assuming that f is constant, prove that the divergence of $\mathbf{v}$ must vanish.

   (b) Prove that the particle paths of a geostrophic flow and its isobars coincide.

   (c) Consider a high pressure field in a geostrophic flow in the northern hemisphere (where $f > 0$). By appealing to the equations in (*) explain whether this high pressure field results in a clockwise or a counterclockwise motion.